Some fascinating developments in mathematics and music

Emmanuel Amiot, France
manu.amiot@free.fr
Associate researcher with LAMPS, University of Perpignan, France
Formerly professor in *Classes préparatoires*

Abstract
The strength of the bonds between music and mathematics goes without saying. This popular belief hides a subtler misconception, that this relationship involves old school mathematics : arithmetics in the Greek School (Pythagoras), diophantine approximations in tuning theory (Euler, Rameau), Fourier series for the decomposition of sound signal, and little else. However, there is much more than that and these two sciences still advance hand in hand as of today. This paper will present by way of example three musical situations involving contemporary mathematical topics : Galois theory in a rhythmic canon problem in the field of minimalist music ; a graph theory question raised by Ludwig van Beethoven which had to wait almost two centuries for an answer ; and a neat word theory theorem discovered in a construction originating in combinations of mystical octaves and fifths in Plato's *Timaeus*.

1 Once upon a time

Two famous quotes seem to circumvent the field of mathematics and music taken together : Pythagoras's "All things are numbers" [1] and Leibniz 's "Musica est exercitium arithmeticae". The latter is closer to modern times but still dates back more than three centuries. It can be surmised that Leibniz already intended "arithmetics" in a broader sense than merely "computing with (integral) numbers", but rather as a sense of symbolic calculation of relationships between objects which is the province of Algebra, [2] especially in view of his own theory of logical calculus, and of the often omitted sequel of the quotation :

> Musica est exercitium arithmeticae *occultum nescientis se numerare animi.* [3]

Whatever he meant, Leibniz certainly knew as a major actor in the field that mathematics had already extended far beyond Ancient Greece's wildest dreams, with whole knew fields of knowledge opening before researchers (analytical geometry, algebraic coordinates, integral and differential calculus. . .). Nowadays, as I am privileged to know in my capacity of Co-Editor in chief of *JMM*, [4] the relationships between maths and music encompass Category Theory, Topology, Graph Theory, Homology, Differential Calculus, Abstract Algebra, Linear Algebra. . . it is actually difficult to find a topic in contemporary mathematics that would be unheard of in music !

The following examples of mathe-musical discoveries do **not** involve numbers, but much more abstract and recent concepts.

2 Tom Johnson's tiling by augmentation

In 2001 during the *Journées d'informatique Musicale* in Bourges, American composer Tom Johnson came up with some examples of a **rhythmic canon** together with a puzzling remark.

A musical canon is made of several voices (singers, or musical instruments, or voices internal to one instrument like Bach's fugas) playing the same motif but at different moments, sometimes allowing

1. We know this through commentators, like Iamblichus's biography and of course Plato's and Aristotle's discourses.
2. The word *algebra* originated in the Xth century in Baghdad with Al Khuwarizmi's magnum opus.
3. "Music is a hidden exercise in arithmetic, of a mind unconscious of dealing with numbers."
4. Journal of Mathematics and Music, Taylor and Francis Eds.

transformations (like playing the motif backwards, or upside down). In this instance, Johnson's motif could be slowed down by 2 (or 4, or 8...) in order to satisfy and additional constraint : *on a given beat there should be exactly one voice playing, no more and no less.* The smallest solution appears in Fig. 1.

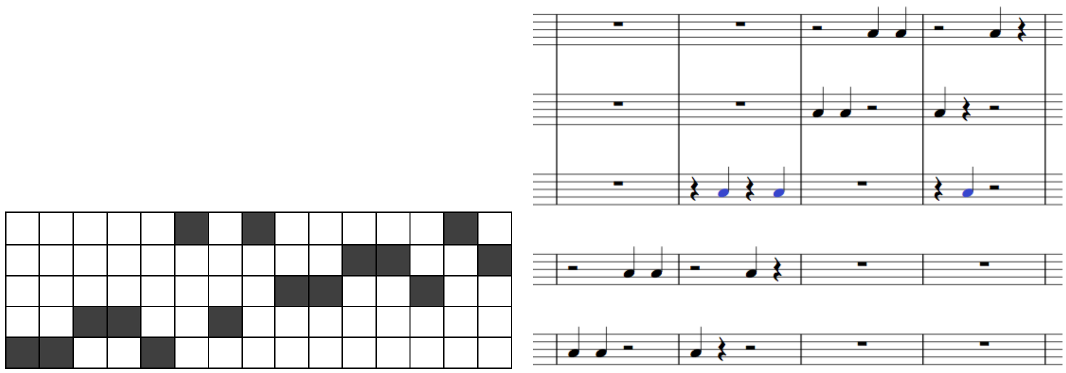


FIGURE 1 – Two representations of Johnson's problem's smallest solution

It is easy to check by trial and error that slowing down (what musicians like Bach called "augmentations") must be used. What puzzled Johnson and mathematician Andranik Tangian, who helped him find all solutions with some maximum length by a brute-force algorithm, was that all solutions had for length a multiple of 15. A solution with three different augmentations and length 30 is shown on Fig. 2.

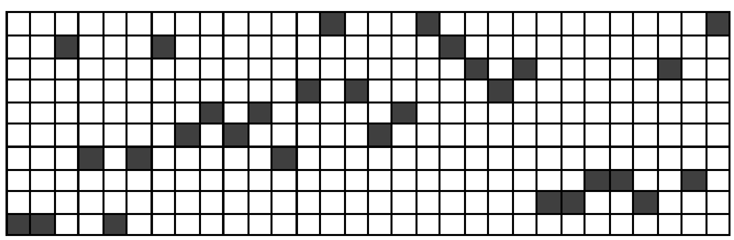


FIGURE 2 – Another solution

For the composer this was amusing and of no consequence : computer-made solutions provided sufficient material for musical pieces. For a mathematician it becomes an irking question : is this a fact ? Can it be proven ?

As far as I know, the only existing proof involves Galois theory of finite fields. Galois devised this theory in solving the open problem of finding formulas for solutions of polynomials equations.[5] His breakthrough consisted in studying symmetries between roots of polynomials – in more general terms, trying to find unavoidable relationships between solutions of a same problem : for instance a multiple root happens when several solutions happen to be identical.

Johnson's problem can be formalized as finding several combinations of transforms of the polynomial $1+X+X^4$ which add up to a sum of consecutive powers. In Fig. 1, this can be expressed[6] as

$$(1+X+X^4)+X^2(1+X+X^4)+X^8(1+X+X^4)+X^{10}(1+X+X^4)+X^5(1+X^2+(X^2)^4) = 1+X+X^2+X^3+\cdots+X^{13}+X^{14}.$$

Now one has to make a huge leap in a fantastic realm where 1+1 is no longer 2, but 0! and where furthermore the aforementioned polynomial $1+X+X^4$ has a root α (it is easy to see that no integer works. No real number, even). This realm exists, it is the so-called Galois field $\mathbb{F}_{16}$ with 16 elements.

5. He proved that there are no formulas in general for equations of degree 5 and higher.

6. This was noted by Tangian, though the translation into polynomial equations of such problems dates back to Hajós and deBruijn in the 1950's.

The astute reader will have noticed that 16 is just one over the mysterious number 15, and this is indeed the key to the proof, because α^{15} is equal to 1. I am afraid I cannot go into the details of a complete mathematical proof, which is developed elsewhere and led to interesting generalizations and original theorems about the existence of tilings and bounds for their sizes, notably enabling some progress on Fuglede's conjecture, a fundamental but still speculative statement connecting different areas of modern mathematics.

3 A hamiltonian graph in Beethoven's ninth symphony

Many believe that graph theory originates with a problem about seven bridges in Königsberg. It is probable instead that the first graph devised by Leonhardt Euler in the years 1730 was the graph of tones, the *Tonnetz* which he published in his 1739 *Tentamen novae theoriae musicae*. In it, tones are connected when they are a fifth or a major third apart. Later versions connect also minor thirds, so that for instance notes C, E and G form a triangle, and a very relevant one since it is a major triad. The other triangles are minor triads as can be seen in Fig. 3.

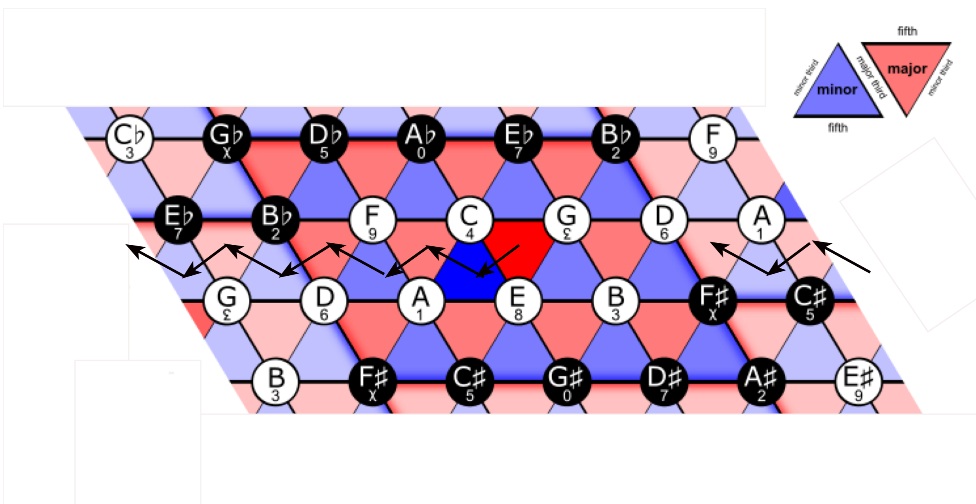


FIGURE 3 – A modern Tonnetz (adapted from Wikipedia)

Around 1820, Beethoven wrote in his IXth symphony a sequence[7] of major and minor triads which are neighbours in the Tonnetz : moving from one to the next involves changing only one note. Such transformations between arbitrary sets of notes were called *parsimonious* by American musicologist Richard Cohn in 1986 (a notable exemple being the modulation from a major scale to its dominant, a fifth apart). He had noticed that Beethoven had found a circuit passing through every one of the 24 minor and major triads, once and only once, and in a parsimonious way.

FIGURE 4 – Beethoven's parsimonious sequence of triads

Such circuits were named *Hamiltonian cycles* in 1856 after the Irish mathematician William R. Hamilton. Although the solution found by Beethoven appears to be quite simple (see the arrows on Fig. 3, the sequence of chords in Fig. 4, and listen for instance to `https://youtu.be/41aLtyd2NOk?t=1212`), the difficulty of finding such cycles cannot be overestimated. Indeed, it has been proven

7. Ninth symphony, second movement, measures 143 to 176.

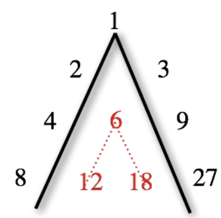
much later that the search for Hamiltonian cycles belongs to the family of the hardest computational problems,[8] those that even the fastest imaginable computers cannot hope to solve.

However, the Tonnetz with its 24 triangles had become tractable in 2009, when two Italian composers inspired by mathematics, Albini and Antonini, managed to find *all* 262 hamiltonian cycles. From this date, several composers used some of these cycles in their musical pieces. An online example is *Aprile*, a pop song written by Moreno Andreatta on a poem by G. d'Annunzio : `https://youtu.be/AB8By7ghTkU`.

It is worth noting that the story does not stop there : not only does this work provide valuable insight into the cycles of a family of graphs, it can be generalized to richer families of musical chords, such as seventh chords, though the computational complexity mentioned above has so far prevented a complete enumeration of all solutions.

4 Palindroms in Λ

In the *Timaeus*, Plato revisited some Pythagorean mystical issues about numbers, notably the Λ-shaped diagram of powers of 2 and 3 and their multiples (see Fig. 5).



1 2 3 4 6 8 9 12 16 18 24 27 32 ...

FIGURE 5 – The original Lambda in *Timaeus*

Notably, some of these values procure frequencies for notes of the scale : starting from (say) F, all powers of 2 are octaves and yield other F's, whereas 9,18,36... are G's, 3, 6, 12... are C's, and so on. What happens in music is that the order of powers (following lines in Fig. 5) is reshuffled for the natural order of integers, from bass to treble. Using combinations of octaves (2) and natural fifth (3/2) as is traditional provides an infinite ascending sequence of notes :

$$\mathsf{F, C, F, G, C, D, F, G, A, C, D, E, F, G, A, B, C, D, E, F, \ldots}$$

Norman Carey had made a study, with his accomplice David Clampitt, of these sequences generated by two non congruent intervals like natural fifth and octave. It was well known for instance that this sequence contains the pentatonic and diatonic scales, and many others. Furthermore, the derived sequence of successive intervals had a few notable features : these intervals get infinitely smaller when the sequence goes on, and when a new interval appears then a previous one disappears never to be seen again, so that at any given time only one of three intervals can appear.[9] The reader may check this on the following labeled sequence of these successive intervals, which N. Carey called *the Lambda word*, after Plato :

$$\Lambda = \mathtt{abcbcdccdcdedcdeddededdeddfddeddfdddfddfdddfddfgfddfddfgfddfgfdfg}$$
$$\mathtt{fddfgfdfgfgfdfgfdfgfgfdfgfgffgfgfdfgfgffgfgfgffgfgffgfgfgffgfgff\mathbf{h}ffgfgffgfgff\mathbf{h}}\ldots$$

wherein in traditional musical terms, $\mathtt{a}$ is a fifth, $\mathtt{b}$ a fourth, $\mathtt{c}$ a whole tone, $\mathtt{d}$ a minor third and so on. Notice that, for instance, the intervals in the pentatonic sequence DFGAC, that is to say $\mathtt{dccd}$,

8. Called *NP-complete* in theoretical computer science.

9. This is essentially Steiner's conjecture, also known as *Three gaps theorem*. For instance see that when $\mathtt{e}$ appears, $\mathtt{c}$ disappears.

form a *palindrom*. The symmetries of these natural musical sequences are well understood. Still it came as a shock to N. Carey when he noticed in June, 2009 that palindroms are *everywhere* in this "word". To quote him,

> Between two adjacent occurrences of any letter, the intervening letters form a palindrom.

On Fig. 5, the two first occurrences of letter `h` have been highlighted, outlining the palindrom `ffgfgffgfgff` between them.

Actually Carey was able to prove a year later (using computational arguments, again a very modern paradigm) that Λ is *saturated* in palindroms, meaning that in a technical sense it is impossible in this type of word to have more palindroms!

This theorem goes beyond the perception of music into the realm of hidden symmetries between harmonics. Word theory (including notions like complexity and palindromicity) is a thriving field, pertaining to the most abstract ideas as well as (obviously) linguistics (with applications like automatic recognition of authorship) or genetics, wherein RNA and DNA are really very long words written with the four letters A, T, C, and G. It is fascinating that, for instance, both genes and musical scales encode geometrical instructions for embedding themselves in multi-dimensional spaces, or that culturally famous scales exhibit special features when considered as words.

To conclude

To me these examples show that further exploration and understanding of music necessitated better honed mathematical instruments. Such discoveries, and even their wording, only became possible with the development of mathematical science : Galois theory on abstract fields for Johnson's canon, word theory in its most recent ventures for Carey's Lambda, graph theory and powerful computers for Beethoven-like harmonic progression. On the other hand, it can be argued that several advances in so-called hard science were made possible by musical queries. Just as it happened time and again with Physics, musical interrogations outside the field of mathematics opened new alleys of thought, helped develop new concepts and eventually unravel original results.